

Modules and Structures of Planar Upper Triangular Rook Monoids

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Abstract

In this paper, we discuss modules and structures of the planar upper triangular rook monoid B_n . We first show that the order of B_n is a Catalan number, then we investigate the properties of a module V over B_n generated by a set of elements v_S indexed by the power set of n . We find that every nonzero submodule of V is cyclic and completely decomposable; we give a necessary and sufficient condition for a submodule of V to be indecomposable. We show that every irreducible submodule of V is 1-dimensional. Furthermore, we give a formula for calculating the dimension of every submodule of V . In particular, we provide a recursive formula for calculating the dimension of the cyclic module generated by v_S , and show that some dimensions are Catalan numbers, giving rise to new combinatorial identities.

Keywords: Rook monoid, order preserving, order decreasing, module, Catalan number, generators and relations.

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1 Introduction

A matrix is a rook matrix if each entry is 0 or 1 and each row and column have at most one 1. A rook matrix A is *planar* or *order preserving* if the matrix obtained from A by deleting all the zero rows and all the zero columns is an identity matrix. The structure and representation theory of the rook monoid, consisting of all rook matrices, are intensively studied [5, 10]. Herbig gives a structure and representation theory of a planar rook monoid [1]. The planar upper triangular rook monoid B_n consists of planar upper triangular rook matrices of size n .

It is natural to ask: What are the representation and structure properties of the planar upper triangular rook monoid? More specifically, how do we construct interesting modules over B_n , and what do irreducible B_n -modules look like? What is the order of B_n and what are the dimensions of the modules of interest? How are the order and the dimensions related to combinatorics? What are the generators and defining relations of B_n ? These questions are closely related to the theory of linear algebraic monoids, since it was made clear in [4, 7] that we are here dealing with the most familiar interesting case of planar upper triangular Renner monoids of reductive monoids. For more information on Renner monoids, see [3, 4, 6, 8, 10].

In this paper we answer the questions above, and our discussion goes a little deeper, showing that the B_n -module properties of V are dramatically different from those of V as

a module over the planar rook monoid. In Section 2 after gathering basic definitions and concepts related to planar upper triangular rook monoids B_n , we give a new interpretation of B_n using generalized reduced echelon matrices. We then calculate in Section 3 the order of B_n in two different ways and show that it is a Catalan number.

Section 4 is devoted to the investigation of B_n -modules over a field F of characteristic 0. Let V_k be a vector space over F generated by a set of elements v_S indexed by the k -subsets of $\mathbf{n} = \{1, \dots, n\}$. Then V_k is a B_n -module under the action (4.1). We are particularly interested in B_n -submodules of V_k and of $V = \bigoplus_{k=0}^n V_k$. We find that every nonzero submodule of V is completely decomposable, and that a submodule of V is indecomposable if and only if it lies in some V_k . Furthermore, we show that every submodule of V is cyclic, and that each irreducible submodule of V is 1-dimensional and is contained in all nonzero submodules of some V_k . We also show that any two different submodules of V are not isomorphic. Moreover, we give a formula for calculating the dimension of every submodule of V using the inclusion-exclusion principle. In particular, we provide a recursive formula for calculating the dimensions of the modules generated by a single basis vector, and find that some of these dimensions are Catalan numbers again, connecting to combinatorics. Viewed as B_t -modules with $t < n$, we are able to decompose some B_n -submodules of V_k into indecomposable B_t -submodules. Section 5 describes the generators and defining relations of B_n .

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2 Preliminaries

Definition 2.1. An injective partial map f of \mathbf{n} is a one-to-one map of a subset $D(f)$ of \mathbf{n} onto a subset $R(f)$ of \mathbf{n} where $D(f)$ is the domain of f and $R(f)$ is the range of f .

We agree that there is a map with empty domain and range and call it 0 map. We can write an injective partial map f of \mathbf{n} in 2-line notation by writing the numbers s_1, \dots, s_k in the top line if $D(f) = \{s_1, \dots, s_k\}$, and then below each number we write its image. Equivalently, we can represent such a map by an $n \times n$ rook matrix, where the entry in the i th row and the j th column is 1 if the map takes j to i , and is 0 otherwise. For example, the map σ given below is an injective partial map of $\mathbf{5}$,

$$\begin{aligned} \sigma &= \begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Definition 2.2. The rook monoid R_n is the monoid of injective partial maps from \mathbf{n} to \mathbf{n} , whose operation is the composition of partial maps and the identity element is the identity map of \mathbf{n} .

Since elements of R_n are not necessarily invertible, R_n is not a group. The map with empty domain and empty range behaves as a zero element. In matrix form, the composition of R_n is consistent with the usual matrix multiplication. Here is an example: for $g = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 5 & 2 \end{pmatrix}$, $f = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix} \in R_5$, we have

$$gf = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 5 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 5 & 2 \end{pmatrix}.$$

The corresponding matrix form of the operation reads as

$$gf = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

An injective partial map from \mathbf{n} to \mathbf{n} is *order preserving* if whenever $a < b$ in the domain of the map, then $f(a) < f(b)$. An injective partial map f is order preserving if and only if the matrix obtained from the matrix form of f by deleting all the zero rows and all the zero columns is an identity matrix; equivalently the graph obtained from the 2-line notation of f by joining all defined $f(a)$ in the range of the map to a is a planar graph, which justifies the name in the following definition.

Definition 2.3. *The planar rook monoid, denoted by PR_n , is the monoid of order preserving injective partial maps from \mathbf{n} to \mathbf{n} .*

Obviously, PR_n is a submonoid of R_n . The structure and representation of the planar rook monoid is studied in Herbig [1]. In particular, V_k is an irreducible PR_n -module.

The next definition will give a different interpretation of an order preserving injective partial map.

Definition 2.4. *A rectangular matrix is a generalized (row and column) reduced echelon matrix if*

- (1) *Each leading entry of a row is 1 and is in a column to the right of the leading entry of the row above it.*
- (2) *Each leading entry of a column is 1 and is in a row below the leading entry of the column to the left of it.*
- (3) *Each leading 1 is the only nonzero entry in its column and its row.*

This definition does not require that all nonzero rows are above any zero rows nor all nonzero columns are to the left of any zero columns. Since the row and column reduced echelon form of a matrix is equivalent to the normal form of the matrix, we can consider a generalized reduced echelon matrix to be a generalization of the normal form of a matrix.

An injective partial map is order preserving if and only if its matrix form is a generalized reduced echelon matrix. Thus, the set of all the generalized reduced echelon

matrices of size n is a monoid with respect to the multiplication of matrices, and the order of this monoid is $\binom{2n}{n}$, since the order of PR_n is

$$|PR_n| = \binom{2n}{n}.$$

An injective partial map is called *order decreasing* if for all a in the domain of the map, we have $f(a) \leq a$. Equivalently, an injective partial map is order decreasing if and only if its matrix form is an upper triangular rook matrix, which motivates the name in the following definition.

Definition 2.5. *The planar upper triangular rook monoid, denoted by B_n , is the monoid of order preserving, order decreasing injective partial maps from \mathbf{n} to \mathbf{n} .*

An injective partial map is in B_n if and only if its matrix form is an upper triangular generalized reduced echelon matrix. In the previous example, we have $f \in B_5$, but $g \notin B_5$.

3 Order of B_n

We first show that the order of B_n is a Catalan number, which is defined by $c_0 = c_1 = 1$ and $c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}$ for $n > 1$ (see [11]).

Proposition 3.1. *Let $n \geq 0$. Then the order of the planar upper triangular rook monoid B_n is the Catalan number c_{n+1} , that is, $b_n = c_{n+1}$.*

Proof. To prove the proposition, we set up a one-to-one correspondence between the set B_n and the set C_{n+1} of all sequences $a_1, a_2, \dots, a_{2n+2}$ of $n+1$ copies of 1's and $n+1$ copies of -1 's, such that $a_1 + a_2 + \dots + a_l \geq 0$ for all $1 \leq l \leq 2n+2$.

Let f be an element of B_n with domain $S = \{s_1 < s_2 < \dots < s_k\}$ and range $T = \{t_1 < t_2 < \dots < t_k\}$. Define $s'_1 = s_1$, $s'_i = s_i - s_{i-1}$ for $2 \leq i \leq k$, and $s'_{k+1} = n+1 - s_k$. Also define $t'_1 = t_1$, $t'_i = t_i - t_{i-1}$ for $2 \leq i \leq k$, and $t'_{k+1} = n+1 - t_k$. Let f' be the sequence

$$\underbrace{1, \dots, 1}_{s'_1}, \underbrace{-1, \dots, -1}_{t'_1}, \underbrace{1, \dots, 1}_{s'_2}, \underbrace{-1, \dots, -1}_{t'_2}, \dots, \underbrace{1, \dots, 1}_{s'_{k+1}}, \underbrace{-1, \dots, -1}_{t'_{k+1}}. \quad (3.1)$$

Now we prove $f' \in C_{n+1}$. By definition of B_n , we have $t_i \leq s_i$ for all $1 \leq i \leq n$. Thus $s'_1 - t'_1 = s_1 - t_1 \geq 0$, $(s'_1 + s'_2 + \dots + s'_i) - (t'_1 + t'_2 + \dots + t'_i) = s_i - t_i \geq 0$ for $2 \leq i \leq k$ and $(s'_1 + s'_2 + \dots + s'_{k+1}) - (t'_1 + t'_2 + \dots + t'_{k+1}) = (n+1) - (n+1) \geq 0$. Denote the partial sum of the first l items in (3.1) by a_l . Then our previous argument shows that $a_l \geq 0$ for $l = s_1 + t_1 + s_2 + t_2 + \dots + s_h + t_h$, where $1 \leq h \leq k+1$, and this implies $a_l \geq 0$ for $1 \leq l \leq 2+2n$ by the format of (3.1). Hence, $f' \in C_{n+1}$.

We next show that the mapping from B_n to C_{n+1} defined by

$$\begin{aligned} \sigma : B_n &\rightarrow C_{n+1} \\ f &\mapsto f' \end{aligned}$$

is bijective. From the definition of the map σ , it is straightforward to see that σ is injective. Now we prove the onto property of the map σ . For arbitrary element $f' = \{a_1, a_2, \dots, a_{2n+2}\}$ of C_{n+1} , the condition $a_1 + a_2 + \dots + a_l \geq 0$ for all $1 \leq l \leq 2n+2$ implies $a_1 = 1$ and $a_{2n+2} = -1$. As in (3.1), let s'_1 be the number of the first consecutive 1's in f' , let t'_1 be the number of the consecutive -1's that follow, and define similarly s'_i and t'_i as before for $2 \leq i \leq k+1$. Let $s_1 = s'_1$, $t_1 = t'_1$, $s_i = s'_1 + s'_2 + \dots + s'_i$, and $t_i = t'_1 + t'_2 + \dots + t'_i$ for $2 \leq i \leq k$. Then the condition $a_1 + a_2 + \dots + a_l \geq 0$ for all $1 \leq l \leq 2n+2$ implies $t_i \leq s_i$ for $1 \leq i \leq k$. Define f to be the mapping $f(s_i) = t_i$ with domain s_1, s_2, \dots, s_k and range t_1, t_2, \dots, t_k . Then $f \in B_n$ and $\sigma(f) = f'$. Hence σ is a one-to-one correspondence between the set B_n and the set C_{n+1} . It is well known that the order of C_{n+1} is the Catalan number c_{n+1} , so the proposition is proved. \square

Remark 3.2. Clearly B_n is a submonoid of the monoid \mathcal{B}_n consisting of all order decreasing (not necessarily planar) injective partial maps. See [2] for the order of \mathcal{B}_n .

The following corollary is immediate.

Corollary 3.3. *The number of upper triangular generalized reduced echelon matrices of size n is the Catalan number c_{n+1} .*

Our next proposition provides a recursive formula for calculating the order b_n of

$$B_n = \{f \in R_n \mid f(j) \leq j \text{ for } j \in D(f); f(i) < f(j) \text{ for } i, j \in D(f) \text{ and } i < j\}.$$

For $0 \leq p, q \leq n-1$, let $b_{p,q} = |B_{p,q}|$ where

$$B_{p,q} = \{f \in B_n \mid D(f) \subseteq \{n-q, \dots, n\} \text{ and } R(f) \subseteq \{n-p, \dots, n\}\}.$$

Proposition 3.4. *Let $n \geq 1$. Then*

- (1) $b_0 = 1, b_1 = 2$.
- (2) $b_{p,0} = p+2$ for $0 \leq p \leq n-1$.
- (3) $b_{p,p} = b_{p+1}$ for $0 \leq p \leq n-1$.
- (4) $b_n = 2b_{n-1} + 1 + \sum_{q=0}^{n-3} b_{n-2,q}$ for $n \geq 2$.
- (5) $b_{p,q} = 1 + \sum_{r=0}^q b_{p-1,r}$ for $1 \leq q < p \leq n-1$.

Proof. Parts (1), (2), and (3) are clear. To prove (4), divide the elements of B_n into two groups: the elements whose ranges contain 1, and those whose ranges do not contain 1. Part (4) follows from the following three identities:

$$\begin{aligned} b_{n-1} &= |\{f \in B_n \mid f(1) = 1\}| = |\{f \in B_n \mid 1 \notin R(f)\}|. \\ b_{n-2,q} &= |\{f \in B_n \mid f(n-1-q) = 1\}| \quad \text{for } 0 \leq q \leq n-3. \\ 1 &= |\{f \in B_n \mid f(n) = 1\}|. \end{aligned}$$

Similarly, for part (5), we divide the elements of $B_{p,q}$ into two groups: the elements whose ranges contain $n-p$, and those whose ranges do not contain $n-p$. Part (5) follows from the following three identities:

$$\begin{aligned} b_{p-1,q} &= |\{f \in B_{p,q} \mid n-p \notin R(f)\}| . \\ b_{p-1,r} &= |\{f \in B_{p,q} \mid f(n-1-r) = n-p\}| \quad \text{for } 0 \leq r \leq q-1 . \\ 1 &= |\{f \in B_{p,q} \mid f(n) = n-p\}| . \end{aligned}$$

□

4 Modules for B_n

A vector space V over a field F of characteristic 0 is called a B_n -module if B_n acts on V satisfying, for all $f, f_1, f_2 \in B_n$, $u, v \in V$, and $\lambda \in F$,

$$\begin{aligned} f \cdot (u + v) &= f \cdot u + f \cdot v, & f_1 \cdot (f_2 \cdot u) &= (f_1 f_2) \cdot u, \\ f \cdot (\lambda u) &= \lambda(f \cdot u), & 1 \cdot u &= u. \end{aligned}$$

From now on, V denotes a vector space with a basis $\mathcal{B} = \{v_S \mid S \subseteq \mathbf{n}\}$ indexed by all the subsets of \mathbf{n} . Then $V = \bigoplus_{S \subseteq \mathbf{n}} Fv_S$ as subspaces is a B_n -module with respect to the following action: for $f \in B_n$ and $S \subseteq \mathbf{n}$,

$$f \cdot v_S = \begin{cases} v_{S'}, & \text{if } S \subseteq D(f) \\ 0, & \text{otherwise,} \end{cases} \quad (4.1)$$

where $S' = \{f(s_1), \dots, f(s_k)\}$ if $S = \{s_1, \dots, s_k\}$. For $0 \leq k \leq n$, let

$$V_k = \text{span}\{v_S \in \mathcal{B} \mid k = |S|\}.$$

Then $V = \bigoplus_{k=0}^n V_k$ is a direct sum of B_n -submodules.

Every module under consideration is a B_n -module over F , unless otherwise stated. Our intent below is to describe the B_n -module structure of V_k and V . To this end, we define a partial order on the power set of \mathbf{n} . For any k -subsets $S = \{s_1 < \dots < s_k\}$ and $T = \{t_1 < \dots < t_k\}$ of \mathbf{n} , define

$$T \leq S \quad \Leftrightarrow \quad t_i \leq s_i \quad \text{for all } i \in \mathbf{k},$$

and a k -subset is not comparable to any l -subset if $k \neq l$.

For $v \in V$ we use $B_n v$ to denote the cyclic submodule of V generated by v . If S is a k -subset of \mathbf{n} , then $B_n v_S$ is a submodule of V_k . Indeed, for any $f \in B_n$ if $S \subseteq D(f)$ then $f(S)$ is a k -subset, so $f \cdot v_S = v_{f(S)} \in V_k$; if S is not a subset of $D(f)$ then $f \cdot v_S = 0 \in V_k$. Some further properties of the module $B_n v_S$ are described in the next result.

Lemma 4.1. *Let S, T be k -subsets of \mathbf{n} .*

- (1) $B_n v_S = \bigoplus_{S' \subseteq \mathbf{n}, S' \leq S} Fv_{S'}$ as vector spaces. In particular, $V_k = B_n v_{\{n-k+1, \dots, n\}}$.
- (2) $B_n v_T \subseteq B_n v_S$ if and only if $T \leq S$.
- (3) $B_n v_S \cap B_n v_T = B_n v_{S \wedge T}$, where $S \wedge T$ is the greatest lower bound of S and T .

Proof. To prove (1) notice that two subsets $S' \leq S$ if and only if $S = D(f)$ and $S' = R(f)$ for a unique $f \in B_n$. Let $S' \leq S$. Then $v_{S'} = f \cdot v_S \in B_n v_S$. Hence $\bigoplus_{S' \subseteq \mathbf{n}, S' \leq S} F v_{S'}$ is included in $B_n v_S$. Conversely, let $x = g \cdot v_S \neq 0$ for some $g \in B_n$. We have $\bar{S} \subseteq D(g)$, $g(S) \leq S$, and hence $x = v_{g(S)} \in \bigoplus_{S' \subseteq \mathbf{n}, S' \leq S} F v_{S'}$. The second part of (i) is now clear.

The proof of (2) follows from (1) since $\{T' \mid T' \subseteq \mathbf{n}, T' \leq T\} \subseteq \{S' \mid S' \subseteq \mathbf{n}, S' \leq S\}$ if and only if $T \leq S$.

To prove (3) let $g \cdot v_S = h \cdot v_T \neq 0$ for some $g, h \in B_n$. Then $g(S) = h(T)$. Suppose

$$S = \{s_1 < \dots < s_k\} \quad \text{and} \quad T = \{t_1 < \dots < t_k\} .$$

Then $S \wedge T = \{\min(s_1, t_1), \dots, \min(s_k, t_k)\}$, and $g(s_i) = h(t_i)$. We define $f \in B_n$ with $D(f) = S \wedge T$ and $R(f) = g(S)$ by $f(\min(s_i, t_i)) = g(s_i)$, where $1 \leq i \leq k$. Then $g \cdot v_S = f \cdot v_{S \wedge T} \in B_n v_{S \wedge T}$, and hence $B_n v_S \cap B_n v_T \subseteq B_n v_{S \wedge T}$. Conversely, for any given $0 \neq f \cdot v_{S \wedge T} \in B_n v_{S \wedge T}$ define $g(s_i) = h(t_i) = f(\min(s_i, t_i))$ for $1 \leq i \leq k$. Then $f \cdot v_{S \wedge T} = g \cdot v_S = h \cdot v_T \in B_n v_S \cap B_n v_T$. The proof of (3) is complete. \square

Let $v = \sum_{S \subseteq \mathbf{n}} \lambda_S v_S$, $\lambda_S \in F$ be a vector of V . The *support* of v is defined to be

$$\text{supp}(v) = \{S \subseteq \mathbf{n} \mid \lambda_S \neq 0\} .$$

Definition 4.2. A vector of the form $w = \sum_{S \in \text{supp}(w)} v_S \in V$ is called a *reduced generator* of a submodule W of V if $W = B_n w$ and W cannot be generated by any other vector whose support contains fewer elements than $\text{supp}(w)$. We agree that 0 is the reduced generator of the zero submodule.

The next proposition gives some properties of submodules of V .

Proposition 4.3. Let $v = \sum_{S \in \text{supp}(v)} \lambda_S v_S \in V$.

- (1) If S is in $\text{supp}(v)$, then $v_S \in B_n v$.
- (2) $B_n v = \bigoplus_{T \in \mathcal{P}(v)} F v_T$ as subspaces, where $\mathcal{P}(v) = \bigcup_{S \in \text{supp}(v)} \{T \subseteq \mathbf{n} \mid T \leq S\}$.
- (3) Every submodule of V is cyclic and contains a unique reduced generator.

Proof. To prove (1) let $\min\{|S| \mid S \in \text{supp}(v)\} = r$. Then there exists an r -subset $T = \{t_1 < \dots < t_r\} \subseteq \mathbf{n}$ such that $T \in \text{supp}(v)$; if $r = 0$, then $T = \emptyset$. Let $f \in B_n$ such that $D(f) = R(f) = T$. By the choice of r , for every $S \in \text{supp}(v)$ with $S \neq T$, there is at least one $s \in S$ such that $s \notin T$, so $f \cdot v_S = 0$. Hence

$$f \cdot v = f \cdot \sum_{S \in \text{supp}(v)} \lambda_S v_S = \sum_{S \in \text{supp}(v)} \lambda_S (f \cdot v_S) = \lambda_T v_T .$$

Thus $v_T \in B_n v$ since $\lambda_T \neq 0$. It is easily seen that

$$\sum_{\substack{S \in \text{supp}(v) \\ |S| > r}} \lambda_S v_S = v - \sum_{\substack{S \in \text{supp}(v) \\ |S| = r}} \lambda_S v_S \in B_n v .$$

Applying the above procedure to $\sum_{S \in \text{supp}(v), |S| > r} \lambda_S v_S$ and iteratively using this procedure, if needed, we get $v_S \in B_n v$ for all $S \in \text{supp}(v)$. The proof of (1) is complete.

From (1) and Lemma 4.1 (1), we have

$$\begin{aligned}
B_n v &= \sum_{S \in \text{supp}(v)} \lambda_S B_n v_S \\
&= \sum_{S \in \text{supp}(v)} \text{span} \{v_T \in \mathcal{B} \mid T \leq S\} \\
&= \bigoplus_{S \in \mathcal{P}(v)} Fv_S, \quad \text{as subspaces.}
\end{aligned}$$

This completes the proof of (2).

We now prove (3). It is trivial for $W = \{0\}$. Let W be a nonzero submodule of V . We claim that W has a basis $\{v_S \in \mathcal{B} \mid S \in \mathcal{P}\}$ for some subset \mathcal{P} of the power set of \mathbf{n} . Indeed, suppose \mathcal{B}_1 is a basis of W and write every element of \mathcal{B}_1 as a linear combination of basis vectors in $\mathcal{B} = \{v_S \mid S \subseteq \mathbf{n}\}$. Let \mathcal{P} be the set of all the different subsets S where S runs through the support of every element of \mathcal{B}_1 . By (1) the set $\{v_S \in \mathcal{B} \mid S \in \mathcal{P}\}$ is a subset of W , and hence a basis of W since it is linearly independent and spans W . Let $w = \sum_{S \in \mathcal{P}} v_S$. By (1) again, W is generated by w , and hence W is cyclic.

We now show how to deduce a reduced generator of W from w . Indeed, if w contains two vectors v_S, v_T with $T \leq S$ and $T \neq S$ in $\text{supp}(w)$, then we can remove the term v_T from w , and by Lemma 4.1 (i) the sum of the remaining terms is still a generator. Repeat this process until we obtain the set

$$\text{Red}(w) = \{S \mid S \text{ is maximal in } \text{supp}(w)\},$$

and then we define the corresponding generator w_{red} of W by

$$w_{\text{red}} = \sum_{S \in \text{Red}(w)} v_S.$$

We claim that w_{red} is a reduced generator of W . Let $v = \sum_{S \in \text{supp}(v)} \lambda_S v_S$ be another generator of W . From Definition 4.2 it suffices to show that $|\text{supp}(v)| \geq |\text{Red}(w)|$. From (2) we find $W = \bigoplus_{T \in \mathcal{P}(v)} Fv_T = \bigoplus_{T \in \mathcal{P}(w)} Fv_T$ where $\mathcal{P}(v)$ and $\mathcal{P}(w)$ are as in (2), and hence $\mathcal{P}(v) = \mathcal{P}(w)$. Define

$$\text{Red}(v) = \{S \mid S \text{ is maximal in } \text{supp}(v)\}. \quad (4.2)$$

Thus, $\text{Red}(v) = \{S \mid S \text{ is maximal in } \mathcal{P}(v)\}$ and $\text{Red}(w) = \{S \mid S \text{ is maximal in } \mathcal{P}(w)\}$. So, $\text{Red}(v) = \text{Red}(w)$ and $|\text{supp}(v)| \geq |\text{Red}(v)| = |\text{Red}(w)|$, showing that w_{red} is reduced.

Suppose that $v = \sum_{S \in \text{supp}(v)} v_S$ is another reduced generator of W . By the definition of reduced generators we know $|\text{supp}(v)| = |\text{Red}(w)|$. Hence $|\text{supp}(v)| = |\text{Red}(v)|$ since $\text{Red}(v) = \text{Red}(w)$. It follows that $\text{supp}(v) = \text{Red}(v)$. Let $v_{\text{red}} = \sum_{S \in \text{Red}(v)} v_S$. Then $v = v_{\text{red}} = w_{\text{red}}$. Therefore w_{red} is the unique reduced generator of W . \square

Definition 4.4. *The set $\text{Red}(v)$ in (4.2) is called the reduced support of v , and the element $v_{\text{red}} = \sum_{S \in \text{Red}(v)} v_S$ is termed the reduced form of v . The reduced support of 0 is empty, and the reduced form of 0 is itself.*

For example, if $n = 7$ and $v = v_\emptyset - 2v_{\{1\}} + v_{\{3\}} + 5v_{\{1,2\}} + 3v_{\{4,7\}} - 2v_{\{5,6\}} + v_{\{1,2,3\}}$, then $\text{Red}(v) = \{\emptyset, \{3\}, \{5, 6\}, \{4, 7\}, \{1, 2, 3\}\}$ is the reduced support of v , and its reduced form is $v_{\text{red}} = v_\emptyset + v_{\{3\}} + v_{\{4,7\}} + v_{\{5,6\}} + v_{\{1,2,3\}}$.

It is sometimes convenient to call the reduced support of v the *reduced support* of the module B_nv . A direct calculation yields that the reduced generator of V_k is $v_{\{n-k+1, \dots, n\}}$ for $1 \leq k \leq n$, and the reduced support of V_k is the set $\{n - k + 1, \dots, n\}$. The module V_0 has the element v_\emptyset as its reduced generator, and its reduced support is the set $\{\emptyset\}$.

The next result is a consequence of Lemma 4.1 (i) and Proposition 4.3 (3).

Corollary 4.5. *If $v, w \in V$, then $B_nv = B_nw$ if and only if they have the same reduced support $\text{Red}(v) = \text{Red}(w)$ if and only if they have the same reduced generator $v_{\text{red}} = w_{\text{red}}$.*

We can now describe the irreducible submodules of V_k for $0 \leq k \leq n$. Write $\mathbf{k} = \{1, \dots, k\}$. If $k = 0$, we agree that $\mathbf{k} = \emptyset$ and $v_{\mathbf{k}} = v_\emptyset$.

Proposition 4.6. *For each $0 \leq k \leq n$, the 1-dimensional submodule $B_nv_{\mathbf{k}}$ is the only irreducible submodule of V_k , and every nonzero submodule of V_k contains $B_nv_{\mathbf{k}}$.*

Proof. Since \mathbf{k} is the smallest element of the set of all k -subsets and the elements of B_n are order decreasing as well as order preserving injective maps, from action (4.1) we find $B_nv_{\mathbf{k}} = Fv_{\mathbf{k}}$ is an irreducible submodule of V_k , and it is 1-dimensional. If W is another nonzero irreducible submodule of V_k , by Proposition 4.3 (3) there exists a generator $w \in V_k$ such that $W = B_nw$. The irreducibility of W forces that $\text{supp}(w)$ contains only the k -subset \mathbf{k} , since if $\text{supp}(w)$ contains another k -subset S different from \mathbf{k} , then by Lemma 4.1 (1), $v_S \in W \setminus B_nv_{\mathbf{k}}$ and hence $B_nv_{\mathbf{k}}$ would be a nonzero proper submodule of W . We conclude $W = B_nv_{\mathbf{k}}$.

Now let W be any nonzero submodule of V_k . From Proposition 4.3 (3) we know that W is generated by a nonzero element $v \in V_k$. Pick any $S \in \text{supp}(v)$. Then $v_S \in W$ by Proposition 4.3 (1). Since there exists a unique map $f \in B_n$ such that $D(f) = S$ and $R(f) = \mathbf{k}$, we find $v_{\mathbf{k}} = f \cdot v_S \in B_nv_S \subseteq W$. Therefore $B_nv_{\mathbf{k}} \subseteq W$. \square

The next result describes irreducible submodules of V in terms of those of V_k .

Proposition 4.7. *If W is an irreducible submodule of V , then $W = B_nv_{\mathbf{k}}$ for some $0 \leq k \leq n$ and $\dim W = 1$. Moreover, $\{B_nv_{\mathbf{k}} \mid k = 0, \dots, n\}$ is a complete set of irreducible submodules of V .*

Proof. From Proposition 4.3 (3) we find $W = B_nw$ for a reduced generator $w \in V$. Since W is nonzero, $\text{supp}(w)$ is not empty. Assume that $S, T \in \text{supp}(w)$ and $S \neq T$. Since S, T are different maximal elements in $\text{supp}(w)$, from Lemma 4.1 (i) we find $v_T \in W \setminus B_nv_S$, and hence B_nv_S is a nonzero proper submodule of W , which contradicts the irreducibility of W . Therefore, $\text{supp}(w)$ contains only one subset of \mathbf{n} , showing that W is a submodule of V_k for some $0 \leq k \leq n$. It follows from Proposition 4.6 that $W = B_nv_{\mathbf{k}}$ and $\dim W = 1$. The second part of the proposition is now straightforward. \square

Recall that a B_n -module is *indecomposable* if it is nonzero and cannot be written as a direct sum of two nonzero submodules, and that a B_n -module is called *completely decomposable* if it is nonzero and is a direct sum of indecomposable submodules.

Proposition 4.8. *Let W be a nonzero submodule of V . Then W is indecomposable if and only if W is a submodule of some V_k where $0 \leq k \leq n$.*

Proof. If W is a nonzero submodule of V_k where $0 \leq k \leq n$, then by Proposition 4.6 any two nonzero submodules of W both contain $B_n v_{\mathbf{k}}$, so their sum cannot be direct, and hence W is indecomposable.

Conversely, if W is a nonzero indecomposable submodule of V , from Proposition 4.3 (3) it follows that $W = B_n v$ for a unique reduced generator $v = \sum_{S \in \text{Red}(v)} v_S \in V$. Let $\mathcal{P}(i)$ be the set of all i -subsets of \mathbf{n} where $0 \leq i \leq n$. For each i let $\text{Red}_i(v) = \text{Red}(v) \cap \mathcal{P}(i)$. Forgetting all the possible empty $\text{Red}_i(v)$, we obtain a partition of

$$\text{Red}(v) = \text{Red}_{i_1}(v) \sqcup \cdots \sqcup \text{Red}_{i_s}(v), \quad \text{for some } 1 \leq s \leq n+1,$$

where $0 \leq i_1 < \cdots < i_s \leq n$. Let $v_{i_j} = \sum_{S \in \text{Red}_{i_j}(v)} v_S$ be the reduced vector with support $\text{Red}_{i_j}(v)$ where $1 \leq j \leq s$. Then

$$W = \bigoplus_{j=1}^s B_n v_{i_j}, \quad \text{direct sum of submodules.} \quad (4.3)$$

Since W is indecomposable and each $B_n v_{i_j}$ is a nonzero proper submodule of W , there exists some $k = i_j$ such that $W = B_n v_k$, which is a submodule of V_k . \square

Corollary 4.9. *Every nonzero submodule of V is completely decomposable. In particular, V is completely decomposable and $V = \bigoplus_{k=0}^n V_k$ is a direct sum of indecomposable submodules.*

Proof. Let W be a nonzero submodule of V . Then from Proposition 4.3 (3) we have $W = B_n v$ for a unique reduced generator $v = \sum_{S \in \text{Red}(v)} v_S \in V$. A similar argument to that of Proposition 4.8 shows that W has the decomposition (4.3). From Proposition 4.8 each $B_n v_{i_j}$ in (4.3) is indecomposable. The first part of the desired result follows. The second part follows immediately. \square

Proposition 4.10. *No two different submodules of V are isomorphic.*

Proof. Let W and U be two submodules of V and $\sigma : W \rightarrow U$ be a module isomorphism. Let $x = \sum_{S \in \mathcal{I}} v_S$ be a reduced generator of W , where \mathcal{I} is an index set. Now for $v_S, S \in \mathcal{I}$, suppose $\sigma(v_S) = \sum_{T \in \mathcal{J}} \lambda_T v_T$ for some index set \mathcal{J} , where $\lambda_T \in F$. Take $f_S \in B_n$ with $D(f) = R(f) = S$. Then

$$\sigma(v_S) = \sigma(f_S \cdot v_S) = f_S \cdot \sigma(v_S) = f_S \cdot \sum_{T \in \mathcal{J}} \lambda_T v_T = \sum_{T \in \mathcal{J}, T \subseteq S} \lambda_T (f_S \cdot v_T) = \sum_{T \in \mathcal{J}, T \subseteq S} \lambda_T v_T.$$

We show that $\sigma(v_S) = \lambda_S v_S$ with $\lambda_S \neq 0$. If in the sum on the right there is some $T' \in \mathcal{J}$, $T' \subsetneq S$ with $\lambda_{T'} \neq 0$, let $f_{T'} \in B_n$ with $D(f_{T'}) = R(f_{T'}) = T'$. We have

$$f_{T'} \cdot \sigma(v_S) = f_{T'} \cdot \sum_{T \in \mathcal{J}, T \subseteq S} \lambda_T v_T = \sum_{T \in \mathcal{J}, T \subseteq S} \lambda_T (f_{T'} \cdot v_T) = \sum_{T \in \mathcal{J}, T \subseteq T'} \lambda_T v_T \neq 0,$$

but $f_{T'} \cdot \sigma(v_S) = \sigma(f_{T'} \cdot v_S) = \sigma(0) = 0$, a contradiction. Thus we get $\sigma(v_S) = \lambda_S v_S$ and $\lambda_S \neq 0$, so $B_n v_S = B_n \sigma(v_S)$. By Proposition 4.3 (1), we find

$$W = B_n x = \sum_{S \in \mathcal{I}} B_n v_S = \sum_{S \in \mathcal{I}} B_n \sigma(v_S) = \sigma \left(\sum_{S \in \mathcal{I}} B_n v_S \right) = \sigma \left(B_n \left(\sum_{S \in \mathcal{I}} v_S \right) \right) = \sigma(B_n x) = U.$$

□

We now describe the dimension of any nonzero submodule of V . Proposition 4.3 (3) assures that the submodule is equal to the module $B_n v$ generated by some $v \in V$.

Proposition 4.11. *Let $v \in V$ and $\text{Red}(v) = \{S_1, \dots, S_m\}$. For any $J \subseteq \text{Red}(v)$ denote by S_J the greatest lower bound of $\{S_j \mid j \in J\}$. Then the dimension of $B_n v$ is given by*

$$\dim B_n v = \sum_{\emptyset \neq J \subseteq \mathbf{m}} (-1)^{|J|-1} \dim B_n v_{S_J}.$$

Proof. From Proposition 4.3 (2) and (3) the dimension of $B_n v$ is equal to the cardinality of the set $\mathcal{P}(v) = \bigcup_{S \in \text{Red}(v)} \{T \subseteq \mathbf{n} \mid T \leq S\}$. Let $A_j = \{T \subseteq \mathbf{n} \mid T \leq S_j\}$, $j \in \mathbf{m}$. Then $\mathcal{P}(v) = \bigcup_{j \in \mathbf{m}} A_j$, and $\dim B_n v_{S_j} = |A_j|$ by Proposition 4.3 (2). With Lemma 4.1 (3) in mind and applying the inclusion-exclusion principle to count the cardinality of $\mathcal{P}(v)$, we obtain the desired formula for $\dim B_n v$. □

We further describe the dimension of $B_n v_S$ for any $S \subseteq \mathbf{n}$. In what follows we agree that if $x > y$, then $\binom{y}{x} = 0$ and the empty sum $\sum_{i=x}^y \square_i = 0$.

Theorem 4.12. *If $S = \{s_1 < \dots < s_k\}$ is a k -subset of \mathbf{n} , let d_k be the dimension of the module $B_n v_S$. We have $d_1 = s_1$, and for $k \geq 2$,*

$$d_k = \sum_{i=1}^{k-1} \binom{s_{k-i+1}}{k+1-i} \gamma_i - \sum_{i=1}^{k-1} \binom{s_{k-i+1} - s_1}{k+1-i} \gamma_i - \sum_{i=1}^{k-2} s_1 \binom{s_{k-i+1} - s_2}{k-i} \gamma_i, \quad (4.4)$$

where $\gamma_1 = 1$ and for $2 \leq j \leq k-1$,

$$\gamma_j = - \sum_{i=1}^{j-2} \binom{s_{k+1-i} - s_{k+2-j}}{j-i} \gamma_i.$$

Proof. By Lemma 4.1 (i) we know that d_k is equal to the number of k -subsets T of \mathbf{n} such that $T \leq S$. Let $\lambda_i = s_{k-i+1} - (k-i+1)$ for $1 \leq i \leq k$. Then $\lambda_i \geq \lambda_{i+1}$ since $s_{k-i+1} > s_{k-i}$. Because the smallest k -subset is $\{1, \dots, k\}$, we have

$$\lambda_1 \geq \dots \geq \lambda_k \geq 0, \quad (4.5)$$

and the number of k -subsets T of \mathbf{n} with $T \leq S$ is equal to the number of all the sequences

$$\mu_1 \geq \cdots \geq \mu_k \geq 0 \quad \text{with} \quad \mu_i \leq \lambda_i \quad \text{for} \quad i = 1, \dots, k. \quad (4.6)$$

To find d_k it suffices to compute the number of the sequences in (4.6) for the given sequence (4.5). If $k = 1$, then $d_1 = \lambda_1 + 1 = s_1$.

If $k \geq 2$, let $2 \leq j \leq k$. For each fixed nonnegative integer $\mu \leq \lambda_j$ we calculate iteratively on j the number $\alpha_j(\mu)$ of the sequences

$$\mu_1 \geq \cdots \geq \mu_{j-1} \geq \mu \quad \text{with} \quad \mu_i \leq \lambda_i \quad \text{for} \quad i = 1, \dots, j-1, \quad (4.7)$$

and the required dimension $d_k = \sum_{\mu=0}^{\lambda_k} \alpha_k(\mu)$.

Let $\xi_j = \lambda_j - \mu$. Then $0 \leq \xi_j \leq \lambda_j$. Our aim now is to prove

$$\alpha_j(\mu) = \beta_j + \gamma_j, \quad (4.8)$$

where $\beta_j = \sum_{i=1}^{j-1} \binom{\lambda_i - \lambda_j + \xi_j + j - i}{j-i} \gamma_i$ and $\gamma_j = -\sum_{i=1}^{j-2} \binom{\lambda_i - \lambda_{j-1} + j - i - 1}{j-i} \gamma_i$ with $\gamma_1 = 1$. Notice that $\alpha_j(\mu)$ is a sum of two numbers β_j and γ_j , of which γ_j depends on $\lambda_1, \dots, \lambda_{j-1}$, whereas β_j depends on $\lambda_1, \dots, \lambda_j$ and ξ_j .

We use induction on j to prove (4.8) for $2 \leq j \leq k$. If $j = 2$, for each fixed nonnegative integer $\mu \leq \lambda_2$ we have $\xi_2 = \lambda_2 - \mu$ and $0 \leq \xi_2 \leq \lambda_2$. Let $\xi_1 = \lambda_1 - \mu_1$. To ensure that (4.7) holds for this case, namely $\mu_1 \geq \mu$ and $\mu_1 \leq \lambda_1$, we must have $0 \leq \xi_1 \leq \lambda_1 - \lambda_2 + \xi_2$, and conversely. So

$$\alpha_2(\mu) = \lambda_1 - \lambda_2 + \xi_2 + 1 = \beta_2 + \gamma_2$$

where $\beta_2 = \lambda_1 - \lambda_2 + \xi_2 + 1$ and $\gamma_2 = 0$, and this is (4.8) for $j = 2$.

Suppose (4.8) holds for $j = l$ with $2 \leq l \leq k-1$, that is, for each fixed nonnegative integer $\mu \leq \lambda_l$ we have $\xi_l = \lambda_l - \mu$ with $0 \leq \xi_l \leq \lambda_l$, and the number of sequences $\mu_1 \geq \cdots \geq \mu_{l-1} \geq \mu$ with $\mu_i \leq \lambda_i$ for $i = 1, \dots, l-1$ is

$$\alpha_l(\mu) = \beta_l + \gamma_l, \quad (4.9)$$

where $\beta_l = \sum_{i=1}^{l-1} \binom{\lambda_i - \lambda_l + \xi_l + l - i}{l-i} \gamma_i$ and $\gamma_l = -\sum_{i=1}^{l-2} \binom{\lambda_i - \lambda_{l-1} + l - i - 1}{l-i} \gamma_i$.

We now prove (4.8) for $j = l+1$. For a fixed nonnegative integer $\nu \leq \lambda_{l+1}$ we have $\xi_{l+1} = \lambda_{l+1} - \nu$ with $0 \leq \xi_{l+1} \leq \lambda_{l+1}$. Let $\mu = \lambda_l - \xi_l$. To ensure that the condition (4.7)

$$\mu_1 \geq \cdots \geq \mu_{l-1} \geq \mu \geq \nu \quad \text{with} \quad \mu_i \leq \lambda_i, \quad i = 1, \dots, l-1 \quad \text{and} \quad \mu \leq \lambda_l$$

holds here, we must have $0 \leq \xi_l \leq \rho_l$ where $\rho_l = \lambda_l - \lambda_{l+1} + \xi_{l+1}$, and conversely. Adding

all $\alpha_l(\mu)$ up for $\nu \leq \mu \leq \lambda_l$ and using the induction hypothesis (4.9), we obtain

$$\begin{aligned}\alpha_{l+1}(\nu) &= \sum_{\mu=\nu}^{\lambda_l} \alpha_l(\mu) = \sum_{\mu=\nu}^{\lambda_l} (\beta_l + \gamma_l) \\ &= \sum_{\xi_l=0}^{\rho_l} \sum_{i=1}^{l-1} \binom{\lambda_i - \lambda_l + \xi_l + l - i}{l - i} \gamma_i + \sum_{\xi_l=0}^{\rho_l} \gamma_l\end{aligned}\quad (4.10)$$

$$\begin{aligned}&= \sum_{i=1}^{l-1} \left\{ \binom{\lambda_i - \lambda_{l+1} + \xi_{l+1} + (l+1) - i}{l+1-i} \gamma_i - \binom{\lambda_i - \lambda_l + l - i}{l+1-i} \gamma_i \right\} \\ &\quad + \binom{\lambda_l - \lambda_{l+1} + \xi_{l+1} + 1}{1} \gamma_l\end{aligned}\quad (4.11)$$

$$\begin{aligned}&= \sum_{i=1}^l \binom{\lambda_i - \lambda_{l+1} + \xi_{l+1} + (l+1) - i}{l+1-i} \gamma_i - \sum_{i=1}^{l-1} \binom{\lambda_i - \lambda_l + l - i}{l+1-i} \gamma_i \\ &= \beta_{l+1} + \gamma_{l+1},\end{aligned}$$

where

$$\begin{aligned}\beta_{l+1} &= \sum_{i=1}^l \binom{\lambda_i - \lambda_{l+1} + \xi_{l+1} + (l+1) - i}{l+1-i} \gamma_i, \\ \gamma_{l+1} &= - \sum_{i=1}^{l-1} \binom{\lambda_i - \lambda_l + l - i}{l+1-i} \gamma_i.\end{aligned}$$

Here we have made use of the identity $\sum_{z=a}^{a+b-1} \binom{z}{p} = \binom{a+b}{p+1} - \binom{a}{p+1}$ in which a, b, p are natural numbers to obtain (4.11) from (4.10) by assigning $a = \lambda_i - \lambda_l + l - i$, $b = \lambda_l - \lambda_{l+1} + \xi_{l+1} + 1$ and $p = l - i \geq 1$. Therefore, (4.8) is valid for $j = l + 1$, and we complete the proof of (4.8) by induction.

We are now able to calculate the dimension d_k of the module $B_n v_S$ for $k \geq 2$ by summing all $\alpha_k(\mu)$ in (4.8) up where μ runs from 0 to λ_k , yielding

$$\begin{aligned}d_k &= \sum_{\mu=0}^{\lambda_k} \alpha_k(\mu) \\ &= \sum_{\xi_k=0}^{\lambda_k} \sum_{i=1}^{k-1} \binom{\lambda_i - \lambda_k + \xi_k + k - i}{k - i} \gamma_i - \sum_{\xi_k=0}^{\lambda_k} \sum_{i=1}^{k-2} \binom{\lambda_i - \lambda_{k-1} + k - i - 1}{k - i} \gamma_i \\ &= \sum_{i=1}^{k-1} \binom{\lambda_i + k - i + 1}{k + 1 - i} \gamma_i - \sum_{i=1}^{k-1} \binom{\lambda_i - \lambda_k + k - i}{k + 1 - i} \gamma_i \\ &\quad - \sum_{i=1}^{k-2} (\lambda_k + 1) \binom{\lambda_i - \lambda_{k-1} + k - i - 1}{k - i} \gamma_i.\end{aligned}$$

From $\lambda_i = s_{k-i+1} - (k - i + 1)$, we conclude that d_k is given by (4.4). \square

The following four corollaries are consequences of Theorem 4.12. Considering the sequence $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ given in (4.5) a partition of $d = \sum_{i=1}^k \lambda_i$ into at most k -parts, we obtain the next result.

Corollary 4.13. *Let $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ be a given partition of some nonnegative integer. Then the number of distinct Young diagrams obtained from the Young diagram of λ by removing zero or more boxes from the rows is d_k , which is given in Theorem 4.12.*

Proof. It is easily seen that the number of distinct Young diagrams obtained from the Young diagram of λ by removing zero or more boxes from the rows is equal to the number of partitions $\mu_1 \geq \cdots \geq \mu_k \geq 0$ such that $\mu_i \leq \lambda_i$ for all $i = 1, \dots, k$. The desired result follows from the proof of Theorem 4.12. \square

Corollary 4.14. *If $S = \{2, 4, \dots, 2k\} \subseteq \mathbf{n}$, then the dimension of the submodule $B_n v_S$ is the Catalan number c_{k+1} .*

Proof. The sequence $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ in (4.5) associated to S is now $k \geq k-1 \geq \cdots \geq 1$. In this case, it is well known that the number of sequences $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0$ such that $\mu_i \leq \lambda_i$ for $1 \leq i \leq k$ is the Catalan number c_{k+1} . The desired result follows from Theorem 4.12. \square

We find a combinatorial identity below for the Catalan number. To our knowledge, the identity is new.

Corollary 4.15. *If $k \geq 2$, then*

$$c_{k+1} = \sum_{i=1}^{k-1} \binom{2(k-i+1)}{k+1-i} \gamma_i - \sum_{i=1}^{k-1} \binom{2(k-i)}{k+1-i} \gamma_i - \sum_{i=1}^{k-2} 2 \binom{2(k-i-1)}{k-i} \gamma_i, \quad (4.12)$$

where $\gamma_1 = 1$ and for $2 \leq i \leq k$,

$$\gamma_i = - \sum_{j=1}^{i-2} \binom{2(i-j-1)}{i-j} \gamma_j. \quad (4.13)$$

Proof. The combinatorial identity is obtained from Theorem 4.12 and Corollary 4.14. \square

Corollary 4.16. *For the k -subset $\{m+1, \dots, m+k\}$ of \mathbf{n} , the dimension of the submodule $B_n v_S$ is $\binom{m+k}{k}$.*

Proof. The sequence in (4.5) corresponding to S is the k -subset $\{m, \dots, m\}$. A direct calculation of d_k for $k \geq 1$ using the formulas given in Theorem 4.12 yields

$$d_k = \binom{m+k}{k}, \quad (4.14)$$

which is the desired result. \square

We now compute the dimension $d_{k,m}$ of the B_n -module $B_n v_{S_{k,m}}$, where $k \geq m$ and

$$S_{k,m} = \{2, 4, \dots, 2m, 2m+1, 2m+2, \dots, 2m+(k-m)\}$$

is a subset of \mathbf{n} , which is a mixture of the two types of subsets in Corollaries 4.14 and 4.16. The sequence (4.5) associated to $S_{k,m}$ is $\{m, m, \dots, m, m-1, m-2, \dots, 1\}$ of length k . Recall that if $x > y$, then $\binom{y}{x} = 0$ and the empty sum $\sum_{i=x}^y \square_i = 0$. Without showing the details, from Theorem 4.12 we obtain, for $m \geq 2$,

$$\begin{aligned} d_{k,m} = & \binom{m+k}{k} - \binom{m+k-2}{k} - 2 \binom{m+k-4}{k-1} + \sum_{i=k-m+3}^{k-1} \binom{2(k-i+1)}{k+1-i} \gamma_i \\ & - \sum_{i=k-m+3}^{k-1} \binom{2(k-i)}{k+1-i} \gamma_i - \sum_{i=k-m+3}^{k-2} 2 \binom{2(k-i-1)}{k-i} \gamma_i \end{aligned} \quad (4.15)$$

where $\gamma_1 = 1, \gamma_2 = \gamma_3 = \dots = \gamma_{k-m+2} = 0, \gamma_{k-m+3} = -1$, and for $i \geq k-m+4$

$$\gamma_i = - \binom{m-k+2i-4}{i-1} - \sum_{j=k-m+3}^{i-2} \binom{2(i-j-1)}{i-j} \gamma_j. \quad (4.16)$$

Notice that $d_{k,k}$ is just the Catalan number c_{k+1} by Corollary (4.14). Formula (4.15) will be used in Corollary 4.20.

Identifying an element of B_t ($t < n$) with an element of B_n that fixes $t+1, \dots, n$, we can regard B_t as a submonoid of B_n , so we are allowed to view any B_n -submodules of V , for example V_k and its submodules, as B_t -modules.

Our aim below is to investigate decompositions of the B_n -module $W_k^m = B_n v_{S_k^m}$ into indecomposable submodules for $S_k^m = \{m+1, \dots, m+k\}$ where k and m are positive integers and $k+m \leq n$. Using the notation above, we obtain the following result.

Proposition 4.17. *Viewed as a B_{m+l} ($1 \leq l < k$) module, $W_k^m = B_n v_{S_k^m}$ is decomposed into a direct sum of indecomposable submodules*

$$W_k^m \downarrow_{B_{m+l}}^{B_{m+k}} \cong \bigoplus_{a=0}^{k-l} \binom{k-l}{a} W_{k-a}^{m+l-k+a}, \quad (4.17)$$

where $\binom{k-l}{a}$ is the multiplicity of the indecomposable submodule $W_{k-a}^{m+l-k+a}$.

Proof. By Lemma 4.1 (1) we have

$$W_k^m = B_{m+k} v_{S_k^m} = \bigoplus_{k\text{-subsets } T \subseteq \{1, \dots, m+k\}} Fv_T.$$

To obtain the desired indecomposable B_{m+l} -submodules on the right of (4.17), we group the 1-dimensional subspaces Fv_T with k -subsets $T \subseteq \{1, \dots, m+k\}$ into categories according to the intersection $\{t'_1, \dots, t'_a\} = T \cap \{m+l+1, \dots, m+k\}$ where $t'_1 < \dots < t'_a$,

$0 \leq a \leq k - l$, and $1 \leq l < k$. Let \mathcal{T} denote the set of all these k -subsets T . For any $T \in \mathcal{T}$, we have

$$T = \{t_1, \dots, t_{k-a}, t'_1, \dots, t'_a\},$$

for some subset $\{t_1, \dots, t_{k-a}\} \subseteq \{1, \dots, m+l\}$ with $t_1 < \dots < t_{k-a}$, so $m+l \geq k-a$.

Write $p = m+l - (k-a)$ and let $T_a = \{p+1, \dots, m+l, t'_1, \dots, t'_a\}$. Then T_a is a k -subset of $\{1, \dots, m+k\}$, and $T \leq T_a$. Define $f \in B_{m+k}$ by

$$f(p+1) = t_1, \dots, f(m+l) = t_{k-a}, f(m+l+1) = m+l+1, \dots, f(m+k) = m+k.$$

Then $T = f(T_a)$. Identifying f with an element of B_{m+l} , we have $v_T = f \cdot v_{T_a} \in B_{m+l}v_{T_a}$. From Lemma 4.1 (i) we have

$$B_{m+l}v_{T_a} = \bigoplus_{T \in \mathcal{T}} Fv_T.$$

We next show that $B_{m+l}v_{T_a} \cong B_{m+l}v_{S_{k-a}^p}$ as B_{m+l} -modules. Notice that

$$S_{k-a}^p = \{p+1, \dots, p+(k-a)\} = \{p+1, \dots, m+l\} \subseteq \{1, \dots, m+l\}.$$

Let $\mathcal{U} = \{U \subseteq \{1, \dots, m+l\} \mid U \leq S_{k-a}^p\}$. By Lemma 4.1 (i) we have

$$B_{m+l}v_{S_{k-a}^p} = \bigoplus_{U \in \mathcal{U}} Fv_U.$$

Since $T_a = S_{k-a}^p \cup \{t'_1, \dots, t'_a\}$, the map of \mathcal{T} to \mathcal{U} defined by

$$T \mapsto T \cap \{1, \dots, m+l\} = \{t_1, \dots, t_{k-a}\}$$

is one-to-one and onto. Since B_{m+l} fixes $\{t'_1, \dots, t'_a\}$ pointwise, the map defined by $v_T \mapsto v_{\{t_1, \dots, t_{k-a}\}}$ leads to a B_{m+l} -module isomorphism of $B_{m+l}v_{T_a}$ onto $B_{m+l}v_{S_{k-a}^p}$, which is indecomposable by Proposition 4.8.

Since there are $\binom{k-l}{a}$ ways to choose $\{t'_1, \dots, t'_a\}$ from $\{m+l+1, \dots, m+k\}$, there are the same number of corresponding modules $B_{m+l}v_{T_a}$ in W_k^m . Thus

$$W_k^m \downarrow_{B_{m+l}}^{B_{m+k}} \cong \bigoplus_{a=0}^{k-l} \binom{k-l}{a} B_{m+l}v_{T_a},$$

and the proof is complete. \square

Corollary 4.18. *Let k, l, m be positive integers with $l < k$.*

$$\binom{m+k}{k} = \sum_{a=0}^{k-l} \binom{k-l}{a} \binom{m+l}{k-a}. \quad (4.18)$$

Proof. Corollary 4.16 shows that $\dim W_k^m = \binom{m+k}{k}$ and $\dim W_{k-a}^{m+l-k+a} = \binom{m+l}{k-a}$. Inserting them into (4.17), we complete the proof. \square

We apply the same procedure in the proof of Proposition 4.17 to deal with, without showing all the details, the decomposition of the B_{2k} -module $W_k = B_n v_{S_k}$ into indecomposable $B_{2(k-1)}$ -modules for $S_k = \{2, 4, \dots, 2k\}$ with $2k \leq n$. We regard W_k as a $B_{2(k-1)}$ -module.

Let S be a k -subset of \mathbf{n} such that $S \leq S_k$. If S contains $2k$, then $v_S = f v_{S_k}$ for some $f \in B_{2(k-1)}$ since $B_{2(k-1)}$ fixes $2k$. Thus the module $B_{2(k-1)} v_{S_k}$ contains basis vectors v_S where S runs through all k -subsets containing $2k$, so $B_{2(k-1)} v_{S_k}$ is isomorphic to the submodule $W_{k-1} = B_{2(k-1)} v_{S_{k-1}}$.

If S does not contain $2k$ but contains $2k-1$, then $S \leq T = \{2, 4, \dots, 2(k-1), 2k-1\}$, and since $B_{2(k-1)}$ fixes $2k-1$, we have $v_S = f v_T$ for some $f \in B_{2(k-1)}$. Thus the module $B_{2(k-1)} v_T$ contains basis vectors v_S where S runs through all k -subsets containing $2k-1$, so it is isomorphic to the submodule $W_{k-1} = B_{2(k-1)} v_{S_{k-1}}$.

If S contains neither $2k$ nor $2k-1$, then $S \leq S_{k,k-2} = \{2, 4, \dots, 2(k-2), 2k-3, 2k-2\}$, and we have $v_S = f v_{S_{k,k-2}}$ for some $f \in B_{2(k-1)}$. The module $B_{2(k-1)} v_{S_{k,k-2}}$ then contains basis vectors v_S where S runs through all k -subsets containing neither $2k$ nor $2k-1$. Therefore, we have shown

Proposition 4.19. *For the k -subset $S_k = \{2, 4, \dots, 2k\}$ of \mathbf{n} , let $W_k = B_n v_{S_k}$. Then viewed as a $B_{2(k-1)}$ module, we have the decomposition of W_k into a direct sum of indecomposable submodules*

$$W_k \downarrow_{B_{2(k-1)}}^{B_{2k}} \cong 2W_{k-1} \oplus B_{2(k-1)} v_{S_{k,k-2}}. \quad (4.19)$$

Corollary 4.20. *Let $k \geq 2$. Then the $(k+1)$ st Catalan number*

$$c_{k+1} = 2c_k + d_{k,k-2}, \quad (4.20)$$

where c_k is the k th Catalan number and $d_{k,k-2}$ is the dimension of $B_{2(k-1)} v_{S_{k,k-2}}$.

Proof. By Corollary 4.14, the dimension of W_k is c_{k+1} and that of W_{k-1} is c_k . The dimension $d_{k,k-2}$ of $B_{2(k-1)} v_{S_{k,k-2}}$ is given in (4.15). Putting them into (4.19), we obtain the desired combinatorial identity (4.20). \square

5 Presentation on generators and relations

We use the method of Section 3 of Herbig [1] to describe generators and relations of B_n . Some preparations are needed. We define a new monoid \hat{B}_n generated by symbols $l_i, \hat{e}_i, \hat{1}$ subject to the relations:

- (i) $\hat{e}_i^2 = \hat{e}_i$
- (ii) $l_i l_{i+1} l_i = l_i l_{i+1} = l_{i+1} l_i l_{i+1}$
- (iii) $l_i \hat{e}_i = l_i = \hat{e}_{i+1} l_i$
- (iv) $l_i \hat{e}_{i+1} = \hat{e}_i \hat{e}_{i+1} = \hat{e}_i l_i = l_i^3 = l_i^2$
- (v) $\hat{e}_i l_j = l_j \hat{e}_i$ for $i \neq j, j+1$

(vi) $l_i l_j = l_j l_i$ for $|i - j| \geq 2$

(vii) $\hat{e}_i \hat{e}_j = \hat{e}_j \hat{e}_i$ for all i, j .

For $a, b \in \mathbf{n}$ with $a > b$, let $L^{a,a} = \hat{1}$ and $L^{a,b} = l_b l_{b+1} \cdots l_{a-2} l_{a-1}$. For any subsets $S = \{s_1 < \cdots < s_k\}$ and $T = \{t_1 < \cdots < t_k\}$ of \mathbf{n} satisfying $s_j \geq t_j$ for all $1 \leq j \leq k$, let

$$U = \mathbf{n} - S = \{u_1 < \cdots < u_{n-k}\} \quad \text{and} \quad V = \mathbf{n} - T = \{v_1 < \cdots < v_{n-k}\}. \quad (5.1)$$

Define $\hat{E}_S = \hat{e}_{u_1} \cdots \hat{e}_{u_{n-k}}$, $L^{S,T} = L^{s_k, t_k} \cdots L^{s_1, t_1}$, and $\hat{E}_T = \hat{e}_{v_1} \cdots \hat{e}_{v_{n-k}}$, where we agree that $\hat{E}_{\mathbf{n}} = \hat{1}$. The word $W_T^S = \hat{E}_T L^{S,T} \hat{E}_S$ is called a *standard word* of \hat{B}_n . The following proposition shows that every element of \hat{B}_n is equivalent, under the relations (i) – (vii), to one of the standard words. First, we note that the generators l_i, \hat{e}_i and $\hat{1}$ are themselves standard words.

Proposition 5.1. (1) $W_T^S l_i = W_{T'}^{S'}$, where

$$(S', T') = \begin{cases} (S, T) & \text{if } i, i+1 \notin S \\ (S \setminus \{i\}, T \setminus \{t_{c+1}\}) & \text{if } i, i+1 \in S \\ ((S \setminus \{i\}) \cup \{i+1\}, T) & \text{if } i \in S, i+1 \notin S \\ (S \setminus \{i+1\}, T \setminus \{t_{c+1}\}) & \text{if } i \notin S, i+1 \in S \end{cases}$$

where $i+1$ is mapped to t_{c+1} under W_T^S .

(2) $W_T^S \hat{e}_i = W_{T'}^{S'}$, where

$$(S', T') = \begin{cases} (S, T) & \text{if } i \notin S \\ (S \setminus \{i\}, T \setminus \{t_c\}) & \text{if } i \in S \end{cases}$$

where i is mapped to t_c under W_T^S .

(3) $l_i W_T^S = W_{T'}^{S'}$, where

$$(S', T') = \begin{cases} (S, T) & \text{if } i, i+1 \notin T \\ (S \setminus \{s_c\}, T \setminus \{i+1\}) & \text{if } i, i+1 \in T \\ (S \setminus \{s_c\}, T \setminus \{i\}) & \text{if } i \in T, i+1 \notin T \\ (S, (T \setminus \{i+1\}) \cup \{i\}) & \text{if } i \notin T, i+1 \in T \end{cases}$$

where s_c is mapped to i under W_T^S .

(4) $\hat{e}_i W_T^S = W_{T'}^{S'}$, where

$$(S', T') = \begin{cases} (S, T) & \text{if } i \notin T \\ (S \setminus \{s_c\}, T \setminus \{i\}) & \text{if } i \in T \end{cases}$$

where s_c is mapped to i under W_T^S .

Proof. We use relations (i) to (vii) repeatedly, but sometimes we do not mention them. We divide the proof of part (1) into four cases.

Case 1.1: neither i nor $i+1$ is in S . Then $i, i+1 \in U$ and \hat{e}_i, \hat{e}_{i+1} appear in \hat{E}_S . Since the items in \hat{E}_S commute with \hat{l}_i except \hat{e}_i, \hat{e}_{i+1} , we have

$$\begin{aligned} W_T^S l_i &= \hat{E}_T \mathbf{L}^{S,T} \hat{e}_{u_1} c \dots \hat{e}_i \hat{e}_{i+1} \dots \hat{e}_{u_{n-k}} l_i \\ &= \hat{E}_T \mathbf{L}^{S,T} \hat{e}_{u_1} \dots \hat{e}_i \hat{e}_{i+1} l_i \dots \hat{e}_{u_{n-k}} \\ &= \hat{E}_T \mathbf{L}^{S,T} \hat{e}_{u_1} \dots \hat{e}_i l_i \dots \hat{e}_{u_{n-k}} \quad (\text{by (iii)}) \\ &= \hat{E}_T \mathbf{L}^{S,T} \hat{e}_{u_1} \dots \hat{e}_i \hat{e}_{i+1} \dots \hat{e}_{u_{n-k}} \quad (\text{by (iv)}) \\ &= W_T^S. \end{aligned}$$

Case 1.2: both i and $i+1$ are in S . We have $i, i+1 \notin U$, and every term in \hat{E}_S commutes with l_i by (v). From (iii) we get

$$\hat{E}_S l_i = \hat{E}_S l_i \hat{e}_i = l_i \hat{E}_S \hat{e}_i = l_i \hat{E}_{S'},$$

where $S' = S \setminus \{i\}$. Let $s_c = i$. Then $s_{c+1} = i+1$ since $i+1 \in S$. For $j < c$, by (vi) the terms in \mathbf{L}^{s_j, t_j} commute with l_i , as their indices are less than or equal to $i-2$. Thus

$$\begin{aligned} W_T^S l_i &= \hat{E}_T \mathbf{L}^{S,T} \hat{E}_S l_i \\ &= \hat{E}_T \mathbf{L}^{S,T} l_i \hat{E}_{S'} \\ &= \hat{E}_T \mathbf{L}^{s_k, t_k} \dots \mathbf{L}^{s_{c+2}, t_{c+2}} (\mathbf{L}^{i+1, t_{c+1}} \mathbf{L}^{i, t_c} l_i) \mathbf{L}^{s_{c-1}, t_{c-1}} \dots \mathbf{L}^{s_1, t_1} \hat{E}_{S'}. \end{aligned} \quad (5.2)$$

We now show

$$\mathbf{L}^{i+1, t_{c+1}} \mathbf{L}^{i, t_c} l_i = L^{i+1, t_c}. \quad (5.3)$$

Indeed, by (vi) the term l_i in $\mathbf{L}^{i+1, t_{c+1}} = l_{t_{c+1}} \dots l_{i-1} l_i$ commutes with all of the terms of $\mathbf{L}^{i, t_c} = l_{t_c} \dots l_{i-2} l_{i-1}$ until l_{i-1} , where we use (ii): $l_i l_{i-1} l_i = l_{i-1} l_i$ to simplify. Repeating the same procedure for each of the remaining terms of $\mathbf{L}^{i+1, t_{c+1}}$, we conclude

$$\begin{aligned} \mathbf{L}^{i+1, t_{c+1}} \mathbf{L}^{i, t_c} l_i &= (l_{t_{c+1}} \dots l_{i-1} l_i) (l_{t_c} \dots l_{i-1}) l_i \\ &= (l_{t_{c+1}} \dots l_{i-1}) (l_{t_c} \dots l_i l_{i-1} l_i) \quad \text{by (vi)} \\ &= (l_{t_{c+1}} \dots l_{i-1}) (l_{t_c} \dots l_{i-1} l_i) \quad \text{by (ii)} \\ &\vdots \\ &= l_{t_c} \dots l_{i-1} l_i \\ &= L^{i+1, t_c}. \end{aligned}$$

Putting (5.3) into (5.2), we obtain

$$\begin{aligned} W_T^S l_i &= \hat{E}_T \mathbf{L}^{s_k, t_k} \dots \mathbf{L}^{s_{c+2}, t_{c+2}} \mathbf{L}^{i+1, t_c} \mathbf{L}^{s_{c-1}, t_{c-1}} \dots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\ &= \hat{E}_T \mathbf{L}^{S', T'} \hat{E}_{S'} \end{aligned} \quad (5.4)$$

where $S' = S \setminus \{i\}$ and $T' = T \setminus \{t_{c+1}\}$.

The right hand side of (5.4) is not a standard word yet because $\hat{e}_{t_{c+1}}$ is still missing in \hat{E}_T . Since $t_c \leq t_{c+1} - 1 \leq i$, the term $l_{(t_{c+1}-1)}$ appears in $\mathbf{L}^{i+1, t_c} = l_{t_c} \cdots l_i$. Notice that $l_{(t_{c+1}-2)}l_{(t_{c+1}-1)} = l_{(t_{c+1}-1)}l_{(t_{c+1}-2)}l_{(t_{c+1}-1)}$ and $\hat{e}_{t_{c+1}}l_{(t_{c+1}-1)} = l_{(t_{c+1}-1)}$. We have

$$\begin{aligned}
L^{i+1, t_c} &= l_{t_c} \cdots l_{(t_{c+1}-3)}(l_{(t_{c+1}-2)}l_{(t_{c+1}-1)}) \cdots l_i \\
&= l_{t_c} \cdots l_{(t_{c+1}-3)}(l_{(t_{c+1}-1)}l_{(t_{c+1}-2)}l_{(t_{c+1}-1)}) \cdots l_i \\
&= l_{t_c} \cdots l_{(t_{c+1}-3)}(\hat{e}_{t_{c+1}}l_{(t_{c+1}-1)})l_{(t_{c+1}-2)}l_{(t_{c+1}-1)} \cdots l_i \\
&= l_{t_c} \cdots l_{(t_{c+1}-3)}\hat{e}_{t_{c+1}}l_{(t_{c+1}-2)}l_{(t_{c+1}-1)} \cdots l_i \\
&= \hat{e}_{t_{c+1}}(l_{t_c} \cdots l_{(t_{c+1}-3)}l_{(t_{c+1}-2)}l_{(t_{c+1}-1)} \cdots l_i) \\
&= \hat{e}_{t_{c+1}}L^{i+1, t_c},
\end{aligned}$$

where we have used that $\hat{e}_{t_{c+1}}$ commutes with all terms on its left. Inserting the above result into (5.4) and knowing that $\hat{e}_{t_{c+1}}$ commutes with $\mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}}$, we deduce

$$\begin{aligned}
W_T^S l_i &= \hat{E}_T(\mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}})(\hat{e}_{t_{c+1}}L^{i+1, t_c})\mathbf{L}^{s_{c-1}, t_{c-1}} \cdots \mathbf{L}^{s_1, t_1}\hat{E}_{S'} \\
&= \hat{E}_T\hat{e}_{t_{c+1}}(\mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}}L^{i+1, t_c}\mathbf{L}^{s_{c-1}, t_{c-1}} \cdots \mathbf{L}^{s_1, t_1})\hat{E}_{S'} \\
&= \hat{E}_{T'}\mathbf{L}^{S', T'}\hat{E}_{S'} \\
&= W_{T'}^{S'}.
\end{aligned}$$

Case 1.3: i is in S , but $i+1$ is not. It follows immediately that $i \notin U, i+1 \in U$, so \hat{e}_{i+1} appears in \hat{E}_S , but \hat{e}_i does not. Using (iii): $\hat{e}_{i+1}l_i = l_i\hat{e}_i$ and (v): $l_i\hat{e}_j = \hat{e}_j l_i$ for $j \neq i, i+1$, we have

$$\begin{aligned}
\hat{E}_S l_i &= \hat{e}_{u_1} \cdots \hat{e}_{i+1} \cdots \hat{e}_{u_{n-k}} l_i \\
&= \hat{e}_{u_1} \cdots \hat{e}_{i+1} l_i \cdots \hat{e}_{u_{n-k}} \\
&= \hat{e}_{u_1} \cdots l_i \hat{e}_i \cdots \hat{e}_{u_{n-k}} \\
&= l_i \hat{E}_{S'},
\end{aligned}$$

where $S' = (S \setminus \{i\}) \cup \{i+1\}$. Let $s_c = i$. Then $s_{c+1} > i+1$ since $i+1 \notin S$. For $j < c$, all the terms in \mathbf{L}^{s_j, t_j} commute with l_i since their indices are at most $i-2$, giving

$$\begin{aligned}
W_T^S l_i &= \hat{E}_T \mathbf{L}^{S, T} \hat{E}_S l_i \\
&= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{i, t_c} \cdots \mathbf{L}^{s_1, t_1} l_i \hat{E}_{S'} \\
&= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots (l_{t_c} \cdots l_{i-2} l_{i-1}) l_i \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\
&= \hat{E}_{T'} \mathbf{L}^{S', T'} \hat{E}_{S'} \\
&= W_{T'}^{S'},
\end{aligned}$$

where $T' = T$.

Case 1.4: $i+1$ is in S , but i is not. We have $i \in U, i+1 \notin U$. The item \hat{e}_i is in \hat{E}_S ,

but \hat{e}_{i+1} is not. By $\hat{e}_i l_i = l_i \hat{e}_{i+1}$, $l_i = l_i \hat{e}_i$, and $l_i \hat{e}_j = \hat{e}_j l_i$ for $j \neq i, i+1$, we find

$$\begin{aligned}
\hat{E}_S l_i &= \hat{e}_{u_1} \cdots \hat{e}_i \cdots \hat{e}_{u_{n-k}} l_i \\
&= \hat{e}_{u_1} \cdots \hat{e}_i l_i \cdots \hat{e}_{u_{n-k}} \\
&= \hat{e}_{u_1} \cdots l_i \hat{e}_{i+1} \cdots \hat{e}_{u_{n-k}} \\
&= \hat{e}_{u_1} \cdots l_i \hat{e}_i \hat{e}_{i+1} \cdots \hat{e}_{u_{n-k}} \\
&= l_i \hat{e}_{u_1} \cdots \hat{e}_i \hat{e}_{i+1} \cdots \hat{e}_{u_{n-k}} \\
&= l_i \hat{E}_{S'} ,
\end{aligned}$$

where $S' = S \setminus \{i+1\}$.

Let $s_{c+1} = i+1$. Then for $j \leq c$, the indices of the terms in \mathbf{L}^{s_j, t_j} are at most $i-2$, so they commute with l_i , leading to

$$\begin{aligned}
W_T^S l_i &= \hat{E}_T \mathbf{L}^{S, T} \hat{E}_S l_i \\
&= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} \mathbf{L}^{i+1, t_{c+1}} \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} l_i \hat{E}_{S'} \\
&= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} (l_{t_{c+1}} \cdots l_{i-1} l_i) l_i \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} .
\end{aligned}$$

Since $l_i^2 = \hat{e}_i \hat{e}_{i+1}$ and $l_{j-1} \hat{e}_j = \hat{e}_{j-1} \hat{e}_j$ for all $t_{c+1}+1 \leq j \leq i$ (use them repeatedly below), we obtain

$$\begin{aligned}
W_T^S l_i &= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} (l_{t_{c+1}} \cdots l_{i-1} l_i^2) \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\
&= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} (l_{t_{c+1}} \cdots l_{i-1} \hat{e}_i \hat{e}_{i+1}) \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\
&= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} (l_{t_{c+1}} \cdots l_{i-2} \hat{e}_{i-1} \hat{e}_i \hat{e}_{i+1}) \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\
&\vdots \\
&= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} (\hat{e}_{t_{c+1}} \cdots \hat{e}_{i-1} \hat{e}_i \hat{e}_{i+1}) \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} .
\end{aligned}$$

As the indices of all the terms l on the left of $\hat{e}_{t_{c+1}}$ are at least $t_{c+1}+1$, it follows from (v) that $\hat{e}_{t_{c+1}}$ commutes with $\mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}}$. We obtain

$$\begin{aligned}
W_T^S l_i &= \hat{E}_T \hat{e}_{t_{c+1}} \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} (\hat{e}_{t_{c+1}+1} \cdots \hat{e}_i \hat{e}_{i+1}) \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\
&= \hat{E}_{T'} \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} (\hat{e}_{t_{c+1}+1} \cdots \hat{e}_i \hat{e}_{i+1}) \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} ,
\end{aligned}$$

where $T' = T \setminus \{t_{c+1}\}$.

Our aim now is to switch \hat{e}_j for $t_{c+1}+1 \leq j \leq i+1$ with the terms l one by one on the right of \hat{e}_j until it encounters either l_j or l_{j-1} , or until it commutes past all of the l . If $t_{c+1}+1 \leq j \leq s_c$, then \hat{e}_j will run into some l_{j-1} in $\mathbf{L}^{s_c, t_c} = l_{t_c} l_{t_c+1} \cdots l_{s_c-1}$, leading to $\hat{e}_j l_{j-1} = l_{j-1}$ by (iii). If $s_c+1 \leq j \leq i+1$, then \hat{e}_j does not run into any l_j or l_{j-1} in \mathbf{L}^{s_c, t_c} since the maximal index of the l there is s_c-1 , and \hat{e}_j does not run into any l_j or l_{j-1} to the right of \mathbf{L}^{s_c, t_c} , either, because all the indices of the l are at most s_c-2 . But for $s_{c+1} \leq j \leq i+1$, we have $j \in \mathbf{n} - S'$, so \hat{e}_j will run into \hat{e}_j in $\hat{E}_{S'}$. By $\hat{e}_j l_{j-1} = l_{j-1}$ and $\hat{e}_j^2 = \hat{e}_j$, we get

$$\begin{aligned}
W_T^S l_i &= \hat{E}_{T'} \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{s_{c+2}, t_{c+2}} \mathbf{L}^{s_c, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\
&= \hat{E}_{T'} \mathbf{L}^{S', T'} \hat{E}_{S'} \\
&= W_{T'}^{S'} .
\end{aligned}$$

This completes the proof of part (1).

We next show part (2). If $i \notin S$, then $i \in U$ and \hat{e}_i is in \hat{E}_S . By $\hat{e}_i^2 = \hat{e}_i$, we have

$$\begin{aligned} W_T^S \hat{e}_i &= \hat{E}_T \mathbf{L}^{S,T} \hat{e}_{u_1} \cdots \hat{e}_i \cdots \hat{e}_{u_{n-k}} \hat{e}_i \\ &= \hat{E}_T \mathbf{L}^{S,T} \hat{e}_{u_1} \cdots \hat{e}_i \hat{e}_i \cdots \hat{e}_{u_{n-k}} \\ &= \hat{E}_T \mathbf{L}^{S,T} \hat{e}_{u_1} \cdots \hat{e}_i \cdots \hat{e}_{u_{n-k}} \\ &= \hat{E}_T \mathbf{L}^{S,T} \hat{E}_S \\ &= W_T^S. \end{aligned}$$

If $i \in S$, let $s_c = i$, and then for $j < c$, all of the indices of the terms in \mathbf{L}^{s_j, t_j} are at most $i - 2$, so they commute with \hat{e}_i . From $\hat{e}_i^2 = \hat{e}_i$, we obtain

$$\begin{aligned} W_T^S \hat{e}_i &= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{i, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_S \hat{e}_i \\ &= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{i, t_c} \cdots \mathbf{L}^{s_1, t_1} \hat{E}_S \hat{e}_i \hat{e}_i \\ &= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots \mathbf{L}^{i, t_c} \hat{e}_i \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} . \end{aligned}$$

where $S' = S \setminus \{i\}$. For $t_c + 1 \leq j \leq i$, using (iv): $l_{j-1} \hat{e}_j = \hat{e}_{j-1} \hat{e}_j$ repeatedly, we get

$$\begin{aligned} W_T^S \hat{e}_i &= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots (l_{t_c} l_{t_c+1} \cdots l_{i-1}) \hat{e}_i \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\ &= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots (l_{t_c} l_{t_c+1} \cdots l_{i-2} \hat{e}_{i-1} \hat{e}_i) \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\ &\vdots \\ &= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots (\hat{e}_{t_c} \hat{e}_{t_c+1} \cdots \hat{e}_{i-1} \hat{e}_i) \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} . \end{aligned}$$

Now the situation is very similar to Case 1.3. We commute \hat{e}_{t_c} past all l on the left and get $\hat{E}_T \hat{e}_{t_c} = \hat{E}_{T'}$, where $T' = T \setminus \{t_c\}$. For $t_c + 1 \leq j \leq s_c - 1$, commute \hat{e}_j past all l on the right until they reach an l_{j-1} , then use $\hat{e}_j l_{j-1} = l_{j-1} \hat{e}_j$. For $s_c - 2 \leq j \leq i$, commute \hat{e}_j past all l on the right and until it meets an \hat{e}_j in $\hat{E}_{S'}$, then use $\hat{e}_j^2 = \hat{e}_j$. We find

$$\begin{aligned} W_T^S \hat{e}_i &= \hat{E}_T \mathbf{L}^{s_k, t_k} \cdots (\hat{e}_{t_c} \cdots \hat{e}_{i-1} \hat{e}_i) \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\ &= \hat{E}_T \hat{e}_{t_c} \mathbf{L}^{s_k, t_k} \cdots (\hat{e}_{t_c+1} \cdots \hat{e}_{i-1} \hat{e}_i) \cdots \mathbf{L}^{s_1, t_1} \hat{E}_{S'} \\ &= \hat{E}_{T'} \mathbf{L}^{S', T'} \hat{E}_{S'} \\ &= W_{T'}^{S'} . \end{aligned}$$

Parts (3) and (4) are similar. □

Remark 5.2. One can prove that the standard words can also be chosen as $\hat{E}_{S,T} \mathbf{L}^{S,T}$ where $\hat{E}_{S,T} = \hat{e}_{w_1} \hat{e}_{w_2} \cdots \hat{e}_{w_h}$ if $\mathbf{n} - S - T = \{w_1, w_2, \dots, w_h\}$, and $\mathbf{L}^{S,T}$ is the same as Proposition 5.1.

Let G be the subset of B_n consisting of the elements 1, l_i , and e_j , where

$$\begin{aligned} l_i &= \begin{pmatrix} 1 & \cdots & i-1 & i+1 & i+2 & \cdots & n \\ 1 & \cdots & i-1 & i & i+2 & \cdots & n \end{pmatrix} \quad \text{and} \\ e_j &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix} , \end{aligned}$$

where $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

Our intention here is to show that G generates B_n . For any arbitrary $a, b \in \mathbf{n}$ with $a > b$, we define $L^{a,b} = l_b \cdots l_{a-2} l_{a-1}$ and $L^{a,a} = 1$. For $f \in B_n$, let

$$S = \{s_1 < \cdots < s_k\} \quad \text{and} \quad T = \{t_1 < \cdots < t_k\}$$

be its respective domain and range. Let U, V be as in (5.1) and let

$$E_S = e_{u_1} \cdots e_{u_{n-k}}, \quad L^{S,T} = L^{s_k, t_k} \cdots L^{s_1, t_1}, \quad E_T = e_{v_1} \cdots e_{v_{n-k}},$$

where we agree that $E_{\mathbf{n}} = 1$. Then $x = E_T L^{S,T} E_S$, and we have shown

Theorem 5.3. *Every element of B_n is a product of elements of G .*

The elements of G satisfy the following relations (we omit the details, which are straightforward), where the indices i and j are such that all expressions in the relations are meaningful.

We use R to denote the set of these relations:

- (1) $e_i^2 = e_i$.
- (2) $l_i l_{i+1} l_i = l_i l_{i+1} = l_{i+1} l_i l_{i+1}$.
- (3) $l_i e_i = l_i = e_{i+1} l_i$.
- (4) $l_i e_{i+1} = e_i e_{i+1} = e_i l_i = l_i^3 = l_i^2$.
- (5) $e_i l_j = l_j e_i$ for $i \neq j, j+1$.
- (6) $l_i l_j = l_j l_i$ for $|i-j| \geq 2$.
- (7) $e_i e_j = e_j e_i$ for all i, j .

Theorem 5.4. *The monoid B_n has presentation $\langle G \mid R \rangle$.*

Proof. The mapping $\phi : \hat{B}_n \rightarrow B_n$ defined by $l_i \mapsto l_i$ and $\hat{e}_i \mapsto e_i$ and $\hat{1} \mapsto 1$ induces a monoid homomorphism of \hat{B}_n onto B_n . It follows from Proposition 5.1 that the mapping is injective. \square

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